Comparison of N-functions

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Abstract

Let $p(t)$ and $q(s)$ be two right-continuous $(s, t > 0)$ nondecreasing functions which are the inverse of one another in the sense that

$$q(s) = \sup_{p(t) \le s} t, \qquad p(t) = \sup_{p(s) \le t} s$$

and which satisfy the conditions $p(0) = q(0) = 0$, $p(+\infty) = q(+\infty) = +\infty$.

A function $M(u)$ is called an N-function if it admits the representation $M(u) = \int_0^{|u|} p(t)\, dt$, where the function $p(t)$ is as mentioned above.

If one of the inequalities $p_1[\alpha q_2(\beta v)] < v$, $p_2[\alpha q_1(\beta v)] > v$, $q_1[\alpha p_2(\beta v)] > u$, $q_2[\alpha p_1(\beta u)] < u$, $p_1[\alpha p_2(\beta u)] > u$ and $p_2[\alpha p_1(u)] > u$ holds for large values u, v and arbitrary constants α, β then $M_1(u) < M_2(u)$. $M_1(u)$ and $M_2(u)$ are equivalent if and only if there exist positive constants k_1, k_2 and u_0 such that $p_1(k_2 u) \le p_2(u) \le p_1(k_1 u)$ $(u \ge u_0)$.

Definitions and Theroems

The theory of N-functions were discussed earlier in Hardy, Littlewood and Pólya [2], Zygmund [3], Krasnosel' skii and Rutickii [4]. In a paper on the comparison of N-functions [4], Kransnosel' skii and Rutickii asserted, with no indication of proof, some properties of N-functions. The purpose of this paper is to give proof of some properties of N-functions. And we shall derive the theorem from the property of N-functions, according to [4, LEMMA 3. 1 p. 17].

If there exist positive constants u_0 and k such that

$$M_1(u) \le M_2(ku) \qquad (u \ge u_0),$$

then we write $M_1(u) < M_2(u)$. And if one of the relations $M_1(u) < M_2(u)$ or $M_2(u) < M_1(u)$ holds, then the N-functions $M_1(u)$ and $M_2(u)$ are said to be comparable.

Let V be a set with elements $u, v, w, \cdots\cdots$. The set V is said to be a partially ordered set by the relation $<$ whenever $u < v$ and $v < w$ implies $u < w$.

Theorem 1. The N-functions form a partially ordered set. That is, $M_1(u) < M_2(u)$ and $M_2(u) < M_3(u)$ implies $M_1(u) < M_3(u)$ for the N-functions $M_1(u)$, $M_2(u)$, $M_3(u)$.

Proof. By hypothesis, $M_1(u) < M_2(u)$ and $M_2(u) < M_3(u)$ holds, so that $M_1(u_1) \le M_2(ku_1)$ $(u_1 \ge u_0 > 0,\ k > 0)$, $M_2(u_2) \le M_3(ku_2)$ $(u_2 \ge u_0 > 0,\ k > 0)$ respectively. In virtue of the fact $M_2(u)$ increases indefinitly, a $u_2 \ge ku_1$ can be found such that $M_2(ku_1) \le M_2(u_2)$. Therefore, we have $M_1(u) < M_3(u)$, from which it follows that $M_1(u) \le M_2(ku)$ for $u \ge u_3 = \max(u_1, u_2) > u_0$ and we may assume that $k > 1$. Thus, the N-functions form a set which is partially ordered relative to the symbol $<$.

We say that the N-functions $M_1(u)$ and $M_2(u)$ are equivalent and write $M_1(u) \sim M_2(u)$ if $M_1(u) < M_2(u)$ and $M_2(u) < M_1(u)$.

Theorem 2. Every N-function $M(u)$ is equivalent to itself.

Proof. The relation $M_1(u) \leqq M_2(ku)$ $(u \geqq u_0,\ k>0)$ implies $M_1(u) < M_2(u)$. If we replace $M_1(u)$ and $M_2(u)$ in the above inequality, then we have $M_2(u) \leqq M_1(ku)$ $(u \geqq u_0,\ k>0)$. This inequality satisfies $M_2(u) < M_1(u)$. In virtue of $M_1(u) < M_2(u)$ and $M_2(u) < M_1(u)$, every N-function $M(u)$ admits the representation $M_1(u) \sim M_2(u) \sim M(u)$.

Theorem 3. The N-functions $M_1(u)$ and $M_2(u)$ are equivalent if and only if there exist positive constants k_1, k_2 and u_0 such that

$$M_1(k_2u) \leqq M_2(u) \leqq M_1(k_1u) \quad (u \geqq u_0) \qquad (*)$$

Proof. By the definition of the equivalence of the N-functios,

$$M_2(u) \leqq M_1(k_1u) \quad (u \geqq u_1 \geqq 0,\ k_1>0) \qquad (1)$$

holds and similarly

$$M_1(u) \leqq M_2(k_3u) \quad (u \geqq u_1>0,\ k_3>0) \text{ holds.} \qquad (2)$$

If we set $k_3u=U$ in (2) then $u=\frac{U}{k_3}$. Hence (2) can be rewritten as

$$M_1(k_2u) \leqq M_2(u), \qquad (3)$$

when we write anew $k_2=\frac{1}{k_3}$ and $U=u$. Now if u_0 is chosen to be the smaller of u_1 and u_2, then it follows from (1) and (3) that $(*)$ whenever $u \geqq u_0$. The sufficiency of the condition $(*)$ is thus proved. Conversely, if $(*)$ holds for $k_1, k_2>0$ and $u \geqq u_0$ then it is clear that $M_1(u) \sim M_2(u)$.

Theorem 4. The N-function $M(u)$ is equivalent to the N function $M(ku)$ for arbitrary $k>0$.

Proof. In virtue of the right inequality of $(*)$, $M_2(u) \leq M_1(\frac{k_1}{k}ku)$. If we assume $\frac{k_1}{k}=m_1>0$, then $M_2(u) \leqq M_1(m_1\ ku)$ $(u \geqq u_0)$. Also in virtue of the left inequality of $(*)$, $M_1(\frac{k_2}{k}ku) \leqq M_2(u)$. We set $\frac{k_2}{k}m_2>0$, then we obtain $M_1(m_2\ ku) \leq M_2(u)$. Therefore, it follows from Theorem 3 that $M(u) \sim M(ku)$ from which it follows that we rewrite $M_1(u) = M_2(u) = M(u)$.

Theorem 5. The N functions $M_1(u)$ and $M_2(u)$ are equivalent if satisfying the condition

$$\lim_{u \to \infty} \frac{M_1(u)}{M_2(u)} = a > 0.$$

Proof. Recalling the definition of the limit, (see Natanson [1]), it is clear that if the limit of the sequence $\{\frac{M_1(u)}{M_2(u)}\}$ is a, then corresponding to any positive number ε, a positive integer u_0 can be assigned such that

$$\left| \frac{M_1(u)}{M_2(u)} - a \right| < \varepsilon, \text{ for all } u>u_0.$$

Then we have $-\varepsilon M_2(u) < M_1(u) - aM_2(u) < \varepsilon M_2(u)$ which can be written as

$$aM_2(u) - \varepsilon M_2(u) < M_1(u) < aM_2(u) + \varepsilon M_2(u).$$

Since $M_1(u)$ and $M_2(u)$ are increase infinitely, we have $M_2(k_2u) \leq M_1(u) \leq M_2(k_1u)$ for $k_1, k_2>0$. Using Theorem 3, we obtain $M_1(u) \sim M_2(u)$.

Theorem 6. If one of the inequalities $p_1[\alpha q_2(\beta v)] < v$, $p_2[\alpha q_1(\beta u)] > v$, $q_1[\alpha p_2(\beta u)] > u$, $q_2[\alpha p_1(\beta u)] < u$, $p_1[\alpha p_2(\beta u)] > u$, $p_2[\alpha p_1(\beta u)] > u$ holds for large values u, v and arbitrary constants α, β then the N-functions satisfy the relation $M_1(u) < M_2(u)$.

Proof. We prove that if $p_1[\alpha q_2(\beta v)] < v$ then $M_1(u) < M_2(u)$. Another assertion can be proved analogously. We set $q_2(\beta v)=u$. Hence we have $\beta v = p_2(u)$. In virtue of the hypothesis, we obtain $p_1[\alpha u] < v$. It follows that $p_1[\alpha u] < v\ \frac{1}{\beta}p_2(u)$. Moreover, if we set $\alpha u=w$ then $u=\frac{w}{\alpha}$. In virtue of the fact that $p_2(u)$ increases indefinitely, a $\frac{w}{\alpha} \geqq u_0$ can be found such that for $u \geqq \frac{w}{\alpha}$ and $k>1$, we have

$$p_1(w) < \frac{1}{\beta}p_2(\frac{w}{\alpha}) \leq p_2(ku).$$

Therefore, using LEMMA 3.1 (see [4] p. 17), we obtain $M_1(u) < M_2(u)$

The next theorem follows from Theorem 3 and LEMMA 3.1 [4, p. 17].

Theorem 7. The N-functions $M_1(u)$ and $M_2(u)$ are equivalent if and only if there exist positive constants k_1, k_2 and u_0 such that

$$p_1(k_2u) \leq p_2(u) \leq p_1(k_1u) \quad (u \geqq u_0).$$

References

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variable ; Ungar, New York, Vol. I (1955), Vol. II (1961).

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N-関数の比較について

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摘　　　要

N-関数 $M(u)$は半順序集合を形成し，自分自身に同値であり，任意の正数 K に対して，$M(ku) \sim M(u)$が成り立つ。

二つの N-関数 $M_1(u)$と $M_2(u)$が同値であるための必要十分条件は，$M_1(k_2u) \leqq M_2(u) \leqq M_1(k_1u) \quad (u \geqq u_0)$ であるか，または $p_1(k_2u) \leqq p_2(u) \leqq p_1(k_1u) \quad (u \geqq u_0)$ であるような正の定数 k_1, k_2と u_0が存在することである。

$$\underset{u \to \infty}{Lim} \frac{M_1(u)}{M_2(u)} = a > 0$$ が成り立つならば $M_1(u) \sim M_2(u)$である。

大きな値 u, v と任意の定数 α, β に対して，$p_1[\alpha q_2(\beta v)] < v$, $p_2[\alpha q_1(\beta v)] > v$, $q_1[\alpha p_2(\beta u)] > u$, $q_2[\alpha p_1(\beta u)] < u$, $p_1[\alpha p_2(\beta u)] > u$, $p_2[\alpha p_1(\beta u)] > u$ の中の一つが成り立てば，$M_1(u) < M_2(u)$である。

正　誤　表

訂正箇所

第Ⅰ部　第12巻第4号

p. 373　Abstract の上から3行目

$q(s)=\sup_{p(t)\leqq s}\leqq t,\quad p(t)=\sup_{p(s)\leqq t}\leqq s$　（誤）

$q(s)=\sup_{p(t)\leqq s} t,\quad p(t)=\sup_{q(s)\leqq t} s$　（正）

同じく Abstract の下から3行目

and　$p_2[\alpha p_1(u)]>u$　（誤）

and　$p_2[\alpha p_1(\beta u)]>u$　（正）

p. 375　右側上から1行目

任意の正数 K に対して　（誤）

任意の正数 k に対して　（正）